The Phenomenology of Superconductivity based on the Ginzburg-Landau Theory

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Introduction

Superconductivity is a phenomenon characterized by two main macroscopic observations: (a) the occurrence of zero resistance and (b) the total or partial expulsion of the magnetic field (Meissner effect) below the superconducting transition temperature (T_c). The Ginzburg-Landau (G-L) theory is a phenomenological theory which initially aimed to explain the essential aspects of superconductivity before a full quantum theory was developed by Bardeen Cooper and Schrieffer around 1956. It is in essence a mean field theory for second order phase transitions, where the symmetry of the system changes discontinuously while its state changes continuously at the transition temperature and is therefore valid for temperatures close to T_c. An order parameter (ψ) is thus defined to describe the ordered phase (superconducting state) and the free energy (F) of the superconductor can be expressed in terms of the order parameter ψ.

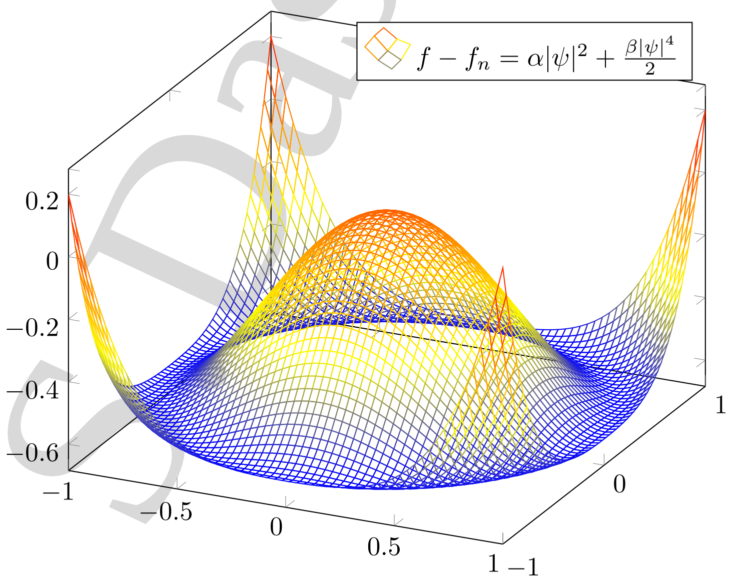
Deriving the G-L equations

In the absence of external magnetic field, the free energy of the superconductor is $F = \int f dV$ where f is the free energy per unit volume and can be written as,

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{1}$$

Here ψ is the order parameter. It is a complex scalar field and α and β are temperature dependent but otherwise constant. The free energy per unit volume of the normal state is f_n.

Plot of the free energy of a superconductor versus order parameter for ($\alpha = -1.3, \beta = 1.4$)



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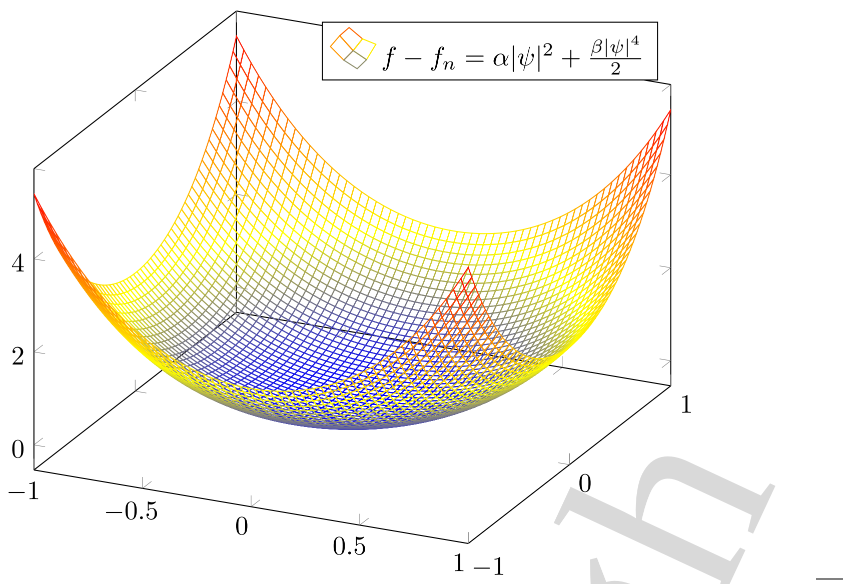
Plot of the free energy versus order parameter for $(\alpha = 1.3, \beta = 1.4)$



In the above figures we depict the dependence of the free energy on the order parameter. For ***positive*** *α the free energy is a paraboloid with minimum at $|\psi| = 0$. On the other hand for α* ***negative****, $f - f_n$ resembles a Mexican hat with a minimum at $|\psi| \neq 0$. The properties of the superconductor depend crucially on α assuming negative values at low temperatures.*

In the presence of external magnetic field , f is given as,

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|\,(\frac{\hbar}{i}\nabla + 2e\vec{A})\psi\,|^2 + \frac{1}{2\mu_0}(\vec{B} - \vec{B_{ext}})^2 \tag{2}$$

Note that is the total field at a point due to the external field and the field generated by the supercurrent. We know that $\vec{B}= \bigtriangledown \times \vec{A}$. On substituting for $\vec{B}$ we obtain,

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|\,(\frac{\hbar}{i}\bigtriangledown + 2e\vec{A})\psi\,|^2 + \frac{1}{2\mu_0}[\,(\bigtriangledown \times \vec{A}) - \vec{B_{ext}}\,]^2 \tag{3}$$

First G.L equation

Minimizing f with respect to ψ^* keeping ψ and $\vec{A}$ fixed yields

$$\delta_{\psi^*} f = \alpha\,\psi\,\delta\psi^* + \beta|\,\psi|^2\,\psi\,\delta\psi^* + \frac{1}{2m^*}[\,(\frac{\hbar}{i}\nabla + 2e\vec{A})\psi\,.\,(\frac{-\hbar}{i}\nabla + 2e\vec{A})\delta\psi*\,] \tag{4}$$

Let us define, $\vec{p} = (\frac{\hbar}{i}\nabla + 2e\vec{A})\psi$. Note that $\vec{p}$ is a vector but, as defined, it is not an operator.

$$\delta_{\psi^*} f = \alpha\,\psi\,\delta\psi^* + \beta\,|\psi|^2\,\psi\,\delta\psi^* + \frac{1}{2m^*}\,[\,\vec{p}\,.\,(\frac{-\hbar}{i}\nabla + 2e\vec{A})\delta\psi^*\,] \tag{5}$$

$$\delta_{\psi^*} f = \alpha\,\psi\,\delta\psi^* + \beta\,|\psi|^2\,\psi\,\delta\psi^* + \frac{1}{2m^*}[\,\frac{-\hbar}{i}\vec{p}.\bigtriangledown\delta\psi* + 2e\vec{p}\,.\,\vec{A}\,\delta\,\psi*\,] \tag{6}$$

We now use the identity,

$$\boldsymbol{\bigtriangledown.(\vec{p}\delta\psi^*) = (\bigtriangledown.\vec{p})\delta\psi^* + \vec{p}.\bigtriangledown\delta\psi^*} \tag{7}$$

Note - We will neglect the $\bigtriangledown.(\vec{p}\delta\psi^*)$ term because when we perform the volume integral and take the limit to infinity, it becomes zero (Refer appendix for the proof).

$$\delta_{\psi*} f = \alpha\,\psi\,\delta\,\psi^* + \beta|\,\psi|^2\,\psi\,\delta\,\psi^* + \frac{1}{2m^*}\,[\,\frac{\hbar}{i}(\bigtriangledown.\vec{p})\delta\psi^* + 2e\vec{p}.\vec{A}\,\delta\psi*\,] \tag{8}$$

on substituting for $\vec{p}$, we get

$$\delta_{\psi*} f = \alpha\,\psi\,\delta\psi^* + \beta\,|\psi|^2\psi\,\delta\psi^* + \frac{1}{2m^*}[\,\frac{\hbar}{i}(\,\bigtriangledown.[\frac{\hbar}{i}\bigtriangledown\psi + 2e\vec{A}\psi]\,)\,+\,2e\,(\frac{\hbar}{i}\bigtriangledown\psi + 2e\vec{A}\psi).\vec{A}\,\,]\delta\psi* \tag{9}$$

$$= \alpha\,\psi\,\delta\psi^* + \beta\,|\psi|^2\,\psi\,\delta\psi^* + \frac{1}{2m^*}[\,-\hbar^2\bigtriangledown^2\psi + \frac{2e\hbar}{i}\bigtriangledown.(\vec{A}\psi) + (\frac{2e\hbar}{i}(\bigtriangledown\psi).\vec{A}) + 4e^2\vec{A}^2\psi\,]\delta\psi* \tag{10}$$

$$= \alpha\,\psi\,\delta\psi^* + \beta\,|\psi|^2\,\psi\;\delta\psi^* + \frac{1}{2m^*}[\,(\frac{\hbar}{i}\,\bigtriangledown + 2e\vec{A})^2\psi\,]\;\delta\psi* \tag{11}$$

Since F has to be an extremum, so that $\delta_{\psi^*}F = 0$ for arbitrary variation of ψ^*, thus:

$$\boldsymbol{\alpha\,\psi + \beta|\psi|^2\psi\; + \frac{1}{2m^*}(\frac{\hbar}{i}\,\bigtriangledown + 2e\vec{A})^2\psi = 0} \tag{12}$$

This equation is known as the "**First G-L Equation**"

Second G.L equation

Minimizing f with respect to A keeping ψ and ψ^* constant yields

$$\delta_A f = \frac{1}{2m^*}\delta_A[\,(\frac{\hbar}{i}\bigtriangledown + 2e\vec{A})\psi.(\frac{-\hbar}{i}\bigtriangledown + 2e\vec{A})\psi^*\,] + \frac{1}{2\mu_0}\delta_A[(\bigtriangledown\times\vec{A}) - \vec{B_{ext}}]^2 \tag{13}$$

We split the evaluation into two parts.

$$\delta_A f = (\delta_A f)_I + (\delta_A f)_{II}$$

The first term in $\delta_A f$ maybe simplified as follows

$$(\delta_A f)_I = \frac{1}{2m^*}\delta_A[\,(\frac{h}{i}\bigtriangledown + 2e\vec{A})\psi.(\frac{-h}{i}\bigtriangledown + 2e\vec{A})\psi^*\,] \tag{14}$$

$$= \frac{1}{2m^*}[\,\frac{2e\hbar}{i}\psi^*\bigtriangledown\psi + 4e^2\vec{A}|\psi|^2 - \frac{2e\hbar}{i}\psi\bigtriangledown\psi^* + 4e^2\vec{A}|\psi|^2]\delta\vec{A}$$

$$= \frac{2e\hbar}{2im^*}\,[\psi^*\bigtriangledown\psi - \psi\bigtriangledown\psi^*].\delta\vec{A} + \frac{4e^2\vec{A}|\psi|^2}{m^*}\,\delta\vec{A} \tag{15}$$

Note that the first term in the above equation is reminescent of the current density in quantum mechanics.

The second term in $\delta_A f$ maybe simplified as follows

$$(\delta_A f)_{II} = \frac{1}{2\mu_0}\delta_A[(\bigtriangledown\times\vec{A}) - \vec{B_{ext}}]^2$$

$$= \frac{1}{2\mu_0}\delta_A[(\bigtriangledown\times\vec{A}).(\bigtriangledown\times\vec{A}) - 2(\bigtriangledown\times\vec{A}).\vec{B}_{ext} + \vec{B}^2_{ext}]$$

$$= \frac{1}{2\mu_0}[(\bigtriangledown\times\vec{A}).(\bigtriangledown\times\delta\vec{A}) + (\bigtriangledown\times\delta\vec{A}).(\bigtriangledown\times\vec{A}) - 2(\bigtriangledown\times\delta\vec{A}).\vec{B}_{ext}]$$

$$= \frac{1}{\mu_0}[(\bigtriangledown\times\vec{A}).(\bigtriangledown\times\delta\vec{A}) - (\bigtriangledown\times\delta\vec{A}).\vec{B}_{ext}]$$

$$= \frac{1}{\mu_0}[(\bigtriangledown\times\vec{A}) - \vec{B}_{ext}].(\bigtriangledown\times\delta\vec{A})$$

$$= \frac{1}{\mu_0}\vec{D}.(\bigtriangledown\times\vec{C})$$

Where $\vec{D}$= $[(\bigtriangledown\times\vec{A}) - \vec{B}_{ext}]$ and $\vec{C} = \delta\vec{A}$

We now employ the vector identity,

$$\boldsymbol{\bigtriangledown.(\vec{C}\times\vec{D}) = \vec{D}.(\bigtriangledown\times\vec{C}) - \vec{C}.(\bigtriangledown\times\vec{D})} \tag{16}$$

$$\Rightarrow \vec{D}.(\bigtriangledown\times\vec{C}) = \bigtriangledown.(\vec{C}\times\vec{D}) + \vec{C}.(\bigtriangledown\times\vec{D})$$

Using this identity in the above expression, we get

$$(\delta_A f)_{II} = \tfrac{1}{\mu_0}[\bigtriangledown.(\delta\vec{A}\times[(\bigtriangledown\times\vec{A}) - \vec{B}_{ext})] + \delta\vec{A}.(\bigtriangledown\times(\bigtriangledown\times\vec{A}) - \bigtriangledown\times\vec{B}_{ext})]$$

At the boundary, we have $\vec{B} = \vec{B}_{ext}$

$$= \frac{1}{\mu_0}\delta\vec{A}.(\bigtriangledown \times (\bigtriangledown \times \vec{A}) - \bigtriangledown \times \vec{B}_{ext})$$

$$= \frac{1}{\mu_0}\bigtriangledown \times (\vec{B} - \vec{B_{ext}}).\delta\vec{A} \tag{17}$$

Summarizing, we have worked out $(\delta_A f)_I$ and $(\delta_A f)_{II}$ Hence we can rewrite $(\delta_A f)$ as,

$$(\delta_A f) = (\delta_A f)_I + (\delta_A f)_{II}$$

$$= \frac{2e\hbar}{2im^*}[\psi^* \bigtriangledown \psi - \psi \bigtriangledown \psi^*].\delta\vec{A} + \frac{4e^2A|\psi|^2}{m^*}.\delta\vec{A} + \frac{1}{\mu_0}\bigtriangledown \times (\vec{B} - \vec{B_{ext}}).\delta\vec{A}$$

If we write

$$\vec{J_s} = \frac{1}{\mu_0}\bigtriangledown \times (\vec{B} - \vec{B_{ext}})$$

we obtain,

$$(\delta_A f) = \frac{2e\hbar}{2im^*}[\psi^* \bigtriangledown \psi - \psi \bigtriangledown \psi^*].\delta\vec{A} + \frac{4e^2A|\psi|^2}{m^*}.\delta\vec{A} + \vec{J_s}.\delta\vec{A} \tag{18}$$

In order to minimize f, equate $(\delta_A f)$ to 0.

$$(\delta_A f) = 0 = \frac{2e\hbar}{2im^*}[\psi^* \bigtriangledown \psi - \psi \bigtriangledown \psi^*].\delta\vec{A} + \frac{4e^2A|\psi|^2}{m^*}.\delta\vec{A} + \vec{J_s}.\delta\vec{A}$$

$$\boldsymbol{\vec{J_s} = (-2e)\frac{\hbar}{2im^*}[\psi^* \bigtriangledown \psi - \psi \bigtriangledown \psi^*] - \frac{4e^2A|\psi|^2}{m^*}} \tag{19}$$

The equation which we just obtained is the "**Second G-L Equation**". To repeat, the first term in the equation is reminescent of the current density except for the prefactor $-2e$ which we hae brackted and which comes from the (Cooper) paired electrons. The second term is also very significant.

Penetration depth

We consder the second G-L equation and attempt its solution. Let $\psi = \sqrt{n_s}\exp(i\phi)$ and this implies $\Rightarrow \psi^* = \sqrt{n_s}\exp(-i\phi)$

On substituting ψ and ψ^* in the above equation for $\vec{J_s}$ we obtain,

$$\vec{J_s} = \frac{ie\hbar}{m^*}[\sqrt{n_s}\exp(-i\phi)\nabla\psi - \sqrt{n_s}\exp(i\phi)\nabla\psi^*] - \frac{4e^2A|\psi|^2}{m^*}$$

$$= \frac{ieh}{m^*}[\sqrt{n_s}\exp(-i\phi).\sqrt{n_s}\exp(i\phi).i.\nabla\phi - \sqrt{n_s}\exp(i\phi)\sqrt{n_s}\exp(-i\phi).(-i).\nabla\phi] - \frac{4e^2A|\psi|^2}{m^*}$$

$$= \frac{ie\hbar}{m^*}[i.n_s\nabla\phi + i.n_s\nabla\phi] - \frac{4e^2An_s}{m^*}$$

$$= \frac{ieh}{m^*}(2in_s)\nabla\phi - \frac{4e^2An_s}{m^*}$$

$$= -\frac{2n_se\hbar}{m_*}(\nabla\phi + \frac{2eA}{h})$$

Taking curl on both sides,

$$\nabla \times \vec{J_s} = -\frac{2n_se\hbar}{m_*}[\nabla \times (\nabla\phi + \frac{2eA}{h})]$$

Substitute $\vec{J_s}$ by $\frac{1}{\mu_0}(\bigtriangledown \times (\vec{B} - \vec{B_{ext}}))$ and noting that $\vec{B_{ext}}$ is constant and hence its curl vanishes we obtain,

$$\bigtriangledown \times (\frac{1}{\mu_0}(\bigtriangledown \times \vec{B})) = -\frac{4n_s e^2 \vec{B}}{m_*}$$

$\Rightarrow$

$$\bigtriangledown^2 \vec{B} = \frac{4n_s e^2 \mu_0}{m^*} \vec{B}$$

Where we have used the following identities

$$\mathbf{1)} \bigtriangledown \times (\bigtriangledown \times \vec{B}) = \bigtriangledown(\bigtriangledown . \vec{B}) - \bigtriangledown^2 \vec{B} \tag{20}$$

We know that $\bigtriangledown . \vec{B} = 0$ Hence

$$\bigtriangledown \times (\bigtriangledown \times \vec{B}) = - \bigtriangledown^2 \vec{B}$$

$$\mathbf{2)} \bigtriangledown \times (\bigtriangledown \phi) = 0 \tag{21}$$

The dimensions of the constant $4n_s e^2 \mu_0 / m^*$ is $1/(\text{length})^2$. We define a length scale λ called the **penetration depth** as

$$\frac{4n_s e^2 \mu_0}{m^*} = \frac{1}{\lambda^2}$$

to obtain

$$\bigtriangledown^2 \vec{B} = \frac{1}{\lambda^2} \vec{B} \tag{22}$$

The equation we have obtained is called the " **London's Equation**". The solution of this equation yields the spatial dependence of the magnetic field.

In one dimension, this reduces to

$$\frac{d^2 B}{dx^2} = \frac{1}{\lambda^2} B \tag{23}$$

The solution of this differential equation is of the form

$$B(x) = B_+ \exp{(\frac{x}{\lambda})} + B_- \exp{(\frac{-x}{\lambda})}$$

We will drop the first term because at infinity it blows up and this is unphysical. Hence the acceptable solution is

$$B(x) = B_- \exp{(\frac{-x}{\lambda})} \tag{24}$$

The variable n_s associated with the order parameter is the number density of Cooper pair electrons. The value of λ is approximately 10 nm for Type I superconductors and 100 nm for Type II superconductors.

Coherence length

Consider the first G-L equation. In the absence of a magnetic field we set $\vec{A}$ to zero. Further, we examine the superconducting phase $T < T_c$ where α is negative. Thus

$$-|\alpha| + \beta|\psi|^2 + \frac{1}{2m^*}(-\hbar^2 \nabla^2)\psi = 0 \tag{25}$$

Rearranging

$$\frac{\hbar^2}{2m^*|\alpha|}\nabla^2\psi + \psi - \frac{\beta}{|\alpha|}|\psi|^2\psi = 0 \tag{26}$$

We define a length scale ξ called the **Coherence Length** by setting $\xi^2 = \hbar^2/2m|\alpha|$ Thus,

$$\xi^2\nabla^2\psi + \psi - \frac{\beta}{|\alpha|}|\psi|^2\psi = 0 \tag{27}$$

We assume that ψ is real and define $f = \psi/\psi_0$ where $\psi_0^2 = |\alpha|/\beta$. Therefore,

$$\xi^2\nabla^2 f + f - f^3 = 0 \tag{28}$$

which in one dimension becomes,

$$\xi^2 \frac{d^2 f}{dx^2} + f - f^3 = 0 \tag{29}$$

Now suppose that there is an interface between a normal metal and a superconductor and that the $y-z$ plane separates the two so that the metal lies in $x<0$ and the superconductor extends through $x>0$. It can be verified that the solution of the above equation subject to the boundary condition that the order parameter vanishes at the boundary is given by:

$$\psi = \psi_0 \tanh(x/\sqrt{2}\xi) \tag{30}$$

This shows that ξ sets the characteristic length scale over which the order parameter has recovered from 0 at the surface to ψ_0 in the bulk and is thus known as the 'coherence length'. For example very close to T_c the parameter α is almost zero. Then ξ is very large which implies that the order parameter $\psi \approx 0$ and superconductivity vanishes.

Note: The coherence length defined above is as per the literature in the field. It is different from the coherence length defined by Pippard and we mention this so that the beginning reader may not be confused

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Flux quantization

We found that in the discussion of the peneteration depth (which was based on the second G-L equation), the supercurrent (J_s) could be expressed as :

$$\vec{J_s} \propto \left(\nabla\phi + \frac{2e\vec{A}}{\hbar}\right) \tag{31}$$

Deep inside the superconductor $J_s = 0$ and hence $\nabla\phi = -2e\vec{A}/\hbar$. Using Stokes' law the magnetic flux,

$$\Phi = \int\int \vec{B}.d\vec{S} = \oint \vec{A}.d\vec{l} = -\frac{\hbar}{2e}\oint \nabla\phi.d\vec{l}$$

As $\psi(\phi) = \psi(\phi + 2n\pi)$ so that ψ remains single valued, when n is an integer. In polar coordinates

$$\nabla\phi = \frac{1}{\rho}\frac{\partial\phi}{\partial\alpha}\hat{\alpha}$$

where ρ is the radial distance and α the polar angle. Further the line element $d\vec{l} = \rho d\alpha\hat{\alpha}$. Hence the net change in the phase $\Delta\phi = \oint \nabla\phi.d\vec{l} = 2\pi n$. Thus,

$$\Phi = \frac{nh}{2e} \tag{32}$$

and flux is quantized.

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Appendix

(Using Gauss Divergence theorem)

$$\int\int\int_V \nabla.(\vec{p}.\delta\psi^*)dv = \oint_S \vec{p}\delta\psi^*.\vec{n}ds$$

Where S is the bounding surface of the superconducting sample. The momentum has no component perpendicular to this surface, so

$$\vec{p}.\vec{n} = 0$$

$$\oint_S \vec{p}\delta\psi^*.\vec{n}ds = 0$$

Hence,

$$\int_V \nabla.(\vec{p}.\delta\psi^*)dv = 0$$

References

Our attempt has been to expose undergraduate students to Ginsburg and Landau's beautiful work on the superconducting phase transition. To this end we have worked out in detail the technical aspects of this work and encourage the student to go forward and explore further. To start with we suggest the chapter on the Landau theory in Michael Tinkham's book "Introduction to Superconductivity" (R. E. Krieger Publishing Company 1980).

