NIUS Lecture Notes

On

Fractals

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NIUS LECTURE NOTES

FRACTALS
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1 Introduction

2 Fractals *

We may have observed the raw fruit of the *kadam* tree (see Fig.(??)), the outline of a cloud or a time-worn, wrinkled face. Or dynamical phenomena such as the zig-zag path of a swimming algae (see Fig. (??)), the growth of polymers, the propagation of cracks, lightning discharge, or the path of a diffusing impurity in a solid. Some have us may have also studied a capacitance transient (see Fig. (??)) or a nuclear decay curve or a first order chemical reaction.

The outlines of such phenomena suggest that these are so jagged that the curves are "long" and at most points "non-differentiable". We examine one such curve, say the coastline of an island. It appears smooth and pearly from far. But as we come closer it appears increasingly irregular. This is one of the characteristics of a fractal curve. The higher the resolution, the more irregular the curve. Often this irregularity has a symmetry paradoxical as this may sound. The outline at a given resolution is similar to the outline at a finer resolution. We shall discuss this self-similar feature of a fractal curve.

A formal procedure to define a fractal proceeds in the following fashion. Consider a straight segment of unit length as shown in Fig. (??). Below it we show three scales or standards denoted by l. The one to the left is of unit length and the other two are of lengths one-half ($l = 1/2$) and and one-third ($l = 1/3$) respectively. We ask ourselves the following question. How many times $N(l)$ do we need to repeat the scale (or standard) so that it spans the straight segment of unit length? The answer is 1, 2 and 3 times respectively with the three scales. It is thus easy to accept the following scaling law for this straight segment,

$$N(l) \sim l^{d_E}$$

where d_E is the Euclidean dimension. In the present case $d_E = 1$. For the area of regular closed curve it is easy to see that d_E would be 2 and similarly 3 for a solid object such as a cube. We can show that for some of the irregular curves mentioned above the scaling law would involve an exponent d_f which is not an integer. We call d_f the fractal dimension. Thus

$$N(l) \sim l^{d_f} \tag{1}$$

We may formalize this by defining a measure μ,

$$\lim_{l \to 0} N(l) l^d = \mu \tag{2}$$

where μ is finite and non-zero at a particular value of d, namely d_f for a fractal curve and d_E for a regular curve.

Construction of a fractal We shall consider a number of examples of fractal curves by actually constructing them.

Example 1: Consider a unit segment. As we approach it the middle third assumes the outline of an equilateral triangle. Alternatively we can say that as our standard l shrinks, it spans a larger length. This process repeats as we come closer and closer or alternatively our standard shrinks further. The Fig.

* All Figures are at the end.

(??) illustrates this process. It has a self-similar character. The figure is called a symmetric Koch curve.

We will calculate the fractal dimension of symmetric Koch curve.
From Fig. (??) we can generate following table (Table (4.1)).

We then have the relationship

$$\begin{aligned} N(l) &\sim l^{d_f} \\ (4)^n &= \left(\frac{1}{3}\right)^{-nd_f} \\ \text{Hence } d_f &= \frac{ln4}{ln3} \end{aligned}$$

Note: Fractal dimension of this curve is between Euclidean dimension 1 and Euclidean dimension 2.

Example 2: We calculate fractal dimension of another curve which in literature goes by the name of Cantor Dust. This is illustrated in Fig. (??).

By looking at the Fig. (??) we arrive at the following table (Table (4.2))

where l, and $N(l)$ have their usual meanings.

$$\begin{aligned} N(l) &= (l)^{-d_f} \\ \text{hence } (2)^n &= \frac{1}{3}^{-nd_f} \\ \text{or } d_f &= \frac{ln2}{ln3} \end{aligned}$$

Note: Fractal dimension of this curve is between Euclidean dimension 0 and Euclidean dimension 1.

Example 3: To calculate fractal dimension of assymetric Koch curve.
This is solved by using the properties of measure μ .

$$\mu = \lim_{l_i \to \infty} \sum N(l_i)\, l_i^{D_o} \tag{3}$$

For $n = 0$

$l_1 = 1$

$N(l_1) = 1$

$$\mu_0 = 1 = \left(\left(\frac{1}{2}\right)^D + 3 \times \left(\frac{1}{4}\right)^D\right)^0$$

For $n = 1$

$l_1 = \frac{1}{2}$; $l_2 = \frac{1}{4}$

$$N(l_1) = 1 \ ; \ N(l_2) = 2$$

$$\mu_1 = 1 = \left(\left(\frac{1}{2}\right)^D + 3 \times \left(\frac{1}{4}\right)^D \right)^1$$

For $n = 2$

$$l_1 = \frac{1}{2} \quad ; \quad l_2 = \frac{1}{4} \quad ; \quad l_3 = \frac{1}{8}$$

$$N(l_1) = 1 \ ; \ N(l_2) = 6 \ ; \ N(l_3) = 9$$

$$\mu_2 = 1 = \left(\left(\frac{1}{2}\right)^D + 3 \times \left(\frac{1}{4}\right)^D \right)^2$$

Similarly.

$$mu_n = 1 = \left(\left(\frac{1}{2}\right)^D + 3 \times \left(\frac{1}{4}\right)^D \right)^n$$

We demand that $\lim_{n\to\infty} \mu_n$ must be finite and non-zero. Therefore

$$\left(\left(\frac{1}{2}\right)^{d_f} + 3 \times \left(\frac{1}{4}\right)^{d_f} \right)^n = 1$$

hence,

$$\left(\frac{1}{2}\right)^{d_f} + 3 \times \left(\frac{1}{4}\right)^{d_f} = 1$$

Define

$$\left(\frac{1}{2}\right)^{d_f} = x;$$

Hence,

$$3x^2 + x - 1 = 0$$

Solving the above quadratic equation we get,

$$x = \frac{-1 + \sqrt{13}}{4}$$

from above we have,

$$d_f = -\frac{lnx}{ln2} = -\frac{ln\frac{-1+\sqrt{13}}{4}}{ln2} \approx 1.2$$

Note: Fractal dimension of this curve is between Euclidean dimension 1 and Euclidean dimension 2.

Remarks Now that we are conversant with how to construct a fractal, a few remarks about them are in order. We note some characteristics of the curves we have constructed. They are infinitely long yet apparently confined to a finite space. As the standard size shrinks they become non-differentiable at more and more points. Finally they are self-similar. Note however that not all fractal curves are self-similar. On a historical note. fractals are not entirely recent. They were known in the late nineteenth century as "monster curves".

An experimental way to arrive at the fractal dimension is to measure $N(l)$ as l shrinks and to plot the logarithm of these quantities. The slope should give us an idea of the fractal dimension. At the end of this chapter there are references which describe a variety of methods to uncover the fractal dimension.

Why do fractals occur in nature? We cannot give a comprehensive answer to this. However the following example may help us. When a cell grows, it divides. This is called cell mitosis. The division occurs since as the cell grows the nutrient requirement of the cell volume increases. But the surface through which the cell gathers its nutrients does not keep pace. In other words the surface to volume ratio shrinks. For larger animals such as humans this problem is even more acute. So our nutrient absorbing organs acquire a complicated "fractal" character. The small intestine is over three meters long, folded and tucked inside our abdomen. Its inner surface has thousands of hair like protrusions called villi, reminiscent of a modern bath towel which has thread-like protrusions and quite efficient at drying us. Our lungs have millions of air sacs called alveoli and its inner surface is indeed large. A human lung can be laid out and stretched over 20 meters! It has also been claimed that the eigenstates of disordered systems has a fractal character.

3 Two Dimensional Shapes

Consider a soap bubble or a bee's honeycomb (see Fig. (??)). There is a pattern which suggest that we study the arrangement of cells in a network. Cellular matter aroused the curiosity of Robert Hooke who in 1660 perfected the microscope and used it to look at cork. He found that it consisted of hexagonal shapes in one plane and box - like shapes in the perpendicular one. He also identified the basic unit of biological structure and called it a "cell".

One can verify the the above laws in a wide variety of situations be it a section of polycrystalline MgO or an arrangement of villages in rural India or even animal habitats. If the points nucleate randomly in space but at the same time and all grow with the same linear growth rate, then the initial structure is a *random Voronoi honeycomb* (two dimension) or a *Voronoi foam* (three dimensions).The cells obviously fill space, and are random. A honeycomb made in this way (Fig. (??)) looks very different from neat hexagons, but it still has the number of sides per face $\bar{n} = 6$ as Euler's law (see below) requires. The nearest thing to Voronoi structures in nature are cellular solids created by the competitive building of sea creatures and of insects: coral and some sponges; the nets of wasps and ants; and ofcourse, the bee's honeycomb. Their great regularity looks quite different from the Voronoi honeycomb of Fig. (??) That is because the creature has a finite size, excluding the nucleation cells from points which are closer than this. This too can be modelled and the result as one might expect, is a foam or a honeycomb of much greater regularity. An example is shown in Fig. (??) in which an array of points are random with the constraints that no two can be closer than a chosen "exclusive distance". The result is not unlike some natural structures (particularly the nest of the wasp). But it still has a slightly angular look that most familiar honeycombs lack. That is because the competitive growth is not the only factor which shapes foams.

There are several others. The most obvious of this is surface tension. When this is the dominant shaping force, then an evenmore regular structure occurs.

From a geometric point of view it is helpful to think of a cellular structure as *vertices*, joined by *edges*, which surround *faces*, which enclose *cells*. (In two dimensions we lose one dimension and think of vertices joined by edges which enclose faces or cells.) The number of edges which meet at a vertex is the edge-connectivity, Z_e (it is usually three in a honeycomb and four in a foam but it can have other values). The number of faces which meet at an edge is the face connectivity, Z_f (usually three for a foam but it, too, can have other values). The number of vertices, V of edges E of faces F and of cells C are related by *Euler's law* which, for a large aggregate of cells, states that:

$$F - E + V = 1 \qquad \text{(two dimensions)} \qquad (4)$$

$$-C + F - E + V = 1 \qquad \text{(three dimensions).} \qquad (5)$$

Fro a general edge co-ordination Z_e, we have,

$$\bar{n} = \frac{2Z_e}{Z_e - 2} \qquad \text{(two dimensions)}$$

which of course reduces to $\bar{n} = 6$, when $Z_e = 3$. Here $\bar{n}$ is the average number of sides per face.

The Aboav-Weaire law and Lewis's rule: We often see that the seven edge cell in a honeycomb has a five edged partner, often as a neighbour. It is generally true that a cell with more sides than average has neighbours which, taken together, have less than the average number. This correlation was noted by Aboav in pictures of honeycombs. The observation is described for honeycombs by Aboav and was given a formal derivation by Weaire:

$$\bar{m} = 5 + \frac{6}{n} \qquad (6)$$

where n is the number of edges of the candidate cell and $\bar{m}$ is the average number of edges of its n neighbours.

The study of cells topology (biological cells this time) has turned up another remarkable result. Lewis, examining a variety of two-dimensional cellular patterns, found that the area of a cell varied linearly with the number of its edges:

$$\frac{A(n)}{A(\bar{n})} = \frac{n - n_0}{\bar{n} - n_0}$$

where $A(n)$ is the area of a cell with the average number of sides, $\bar{n}$, and n_0 is a constant (Lewis finds $n_0 = 2$).This rule holds for Voronoi cells; Lewis finds that it holds for most other two dimensional cells as well.

4 3-Dimensional Structures

What kind of disorders do you expect? How do you characterize them?

I : SUBSTITUTIONAL DISORDER
Lattice exists but atoms of two species occupy them at random
Example: $Cu_{1-x}Ni_x$ on an f.c.c lattice
$GaAs_{1-x}Sb_x$ A zinc blended lattice.
The question of characterizing them is linked with the question," What kind of theory you want to do?"
For CPA (Effective Medium Theory) you need.

(i) Underlying Xtal structure.

(ii) Lattice constant.

(iii) The concentrations of Cu and Ni (A and B atoms) C_A & C_B

$$C_A + C_B = 1$$

These are the three structural parameters you need.

II : LIQUID LIKE DISORDER
Consider liquid metal. Most metals in solid state have f.c.c structure. This is a

(i) close packed structure with packing fraction$\simeq 0.74$.

(ii) $n.n. = 12$

It is believed that the liquid state metal also must resemble the solid. This idea was pioneered by Besnal who proposed a Deuse Random Packing of Hard Spheres (DRPHS). The Bernal School's effort reached titanic proportions with Finney(1970)

Found: packing fraction 0.64
$n.n. \simeq 8$ (can be stretched to 12, like beauty it all lies in the eye of the beholder)
But in a real liquid metal, packing fraction$\simeq 0.74$
$\therefore$ Opinion was turned against this DRPHS until it was realized that soft spheres should be used. This rectified matters.
Bennet(1972) obtained similar results by resorting to a computer algorithm.
The important parameter is the radial or pair distribution function.

$$g(R) = g(|R|)$$

How to characterize these materials. Again it depends on the type of theory employed. For an effective medium theory

(i) n : # density of ions (atoms)

(ii) $g(R)$ pdf

Applications:

(i) Liquid Metals

(ii) Metallic Glass

III : TOPOLOGICAL DISORDER

$g(\underline{R})$: depends on $\underline{R}$

Angular Dep.

Ex. $\alpha - Si$: Bond length = Xtalline bond length
Bond Lle = Xtalline bond Lle
Dihedral Lle = 10% diafirtion

Dihedral Lle = 60^0 in diamond / zinc-blende
= 0^o wurtite.

No good effective medium theory, since no one has a handle on how to characterize the disorder. Hence the popularity of Bethe lattice Approaches

EXERCISE 4

1. Read the passage and answer the questions based on it.

In his varied wanderings Budhuram chanced upon a huge medieval church in a small Italian town. The church had a fascinating history. It was a massive complex structure the likes of which he had not seen even in Rome. He learnt that during construction the roof had *collapsed twice* and was eventually built by criminals condemned to death. Its central hall was very long and had massive pillars. Budhuram had seen similar pillars in the temples of his native Hindoostan and knew the reasons for them. In fact he had a simple scaling argument for their existence. Budhuram also noticed that the church had long stained glass windows. But these were not the only interesting features. As he walked down what he thought was a long straight hall he espied a square room to the side. This room had a similar smaller square room attached which in turn had other smaller square rooms.

The whole church had a labyrinthine of such rooms. He sketched a part of it. Being adept he quickly calculated the (fractal) dimension of the "monster" curve suggested by his sketch. After all in his travels he had familiarized himself with the work of Weierstrass and his disciples in this field. But he had no explanation for their existence.

In the evening he returned to the inn where he was lodged. He sat for dinner in the bar attached to the inn. And brooded over the existence of the large windows and the labyrinthine structure of the church. What motives could he attach to the builders of this strange structure? He thought of his mentor, the venerable VAS of Cawnpore. VAS had frequently admonished him on being quick in mathematics but dim-witted in the natural sciences. Perhaps he was right. "**SEEK DISORDER, SEE DISORDER**", the venerable, worldly wise, and weary VAS had exhorted. That was one reason Budhuram had taken to travel and was staying in the squalid inn instead of the serene monastery nearby. A drunken brawl broke out in the bar. This was too much for the "order-minded" Budhuram. He fled to his room upstairs. The room was small, damp and dark. He lit a candle. As the brightness suffused the room, Budhuram sat back and smiled. He had chanced upon the answers to the questions which had intrigued him all day. [***The above passage is taken from a text discovered from***

a dilapidated structure near the ruins of Nalanda Monastery (University). It is dated circa nineteenth century

Questions

1. What was the fractal dimension of the church?
2. Can you guess the scaling argument for the existence of the massive pillars which Budhuram seem to know so well?
3. Why did the church have a "fractal" structure?

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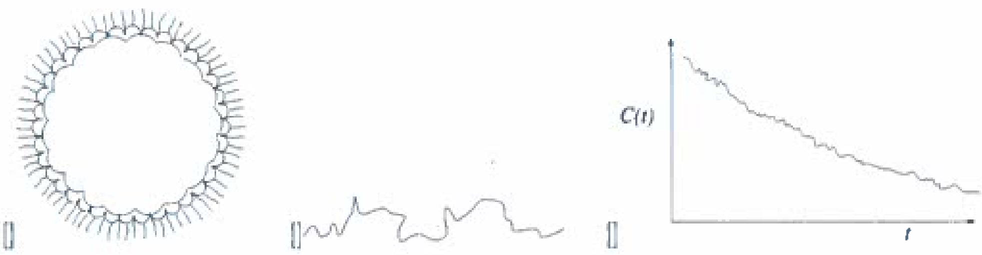


Figure 1: (a) The outline of the raw fruit of *kadam*, (b) The path of a swimming algae, (c)The capacitance transient or nuclear decay curve.

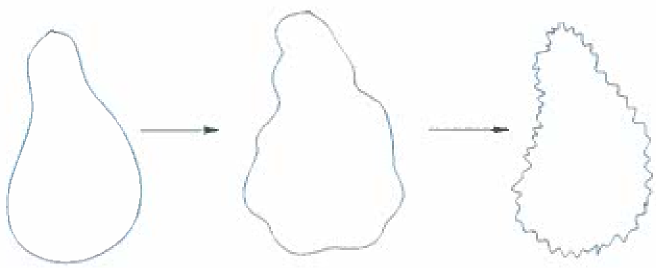


Figure 2: Coastlines viewed as we come closer.

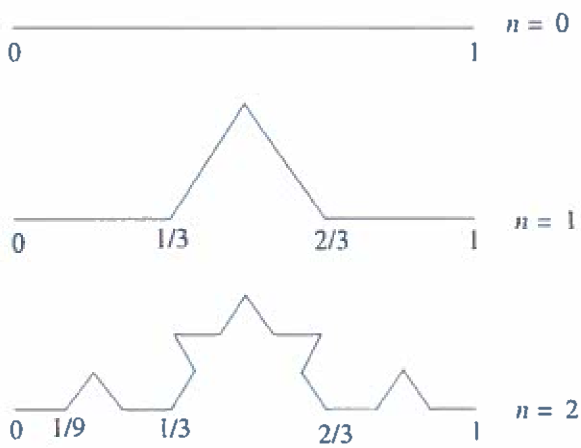


Figure 3: The Koch curve

Table 1: The scale l decreases successively as $N(l)$ increases. The Table is for the Koch curve.

No. of iterations (n)	l	$N(l)$
0	1	1
1	1/3	4
2	1/9	16
3	1/27	64
4	1/81	256

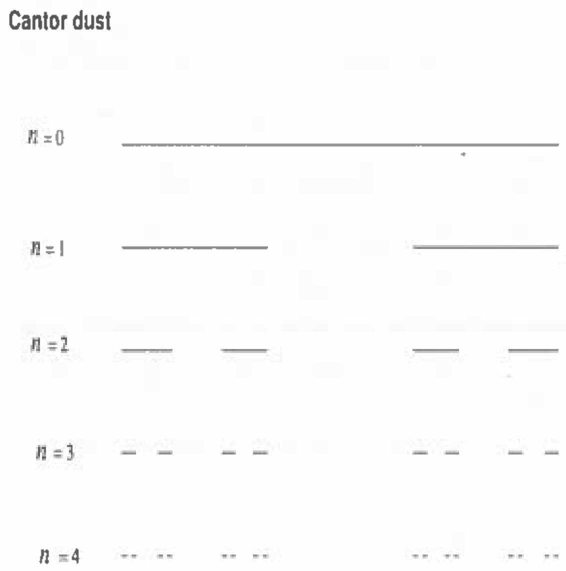


Figure 4: The Cantor Dust

Table 2: As the scale l decreases, $N(l)$ increases.

No. of iterations (n)	l	$N(l)$
0	1	1
1	1/3	2
2	1/9	4
3	1/27	8

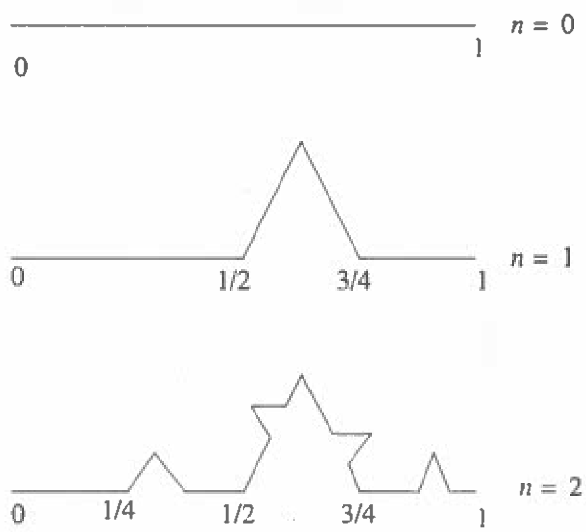


Figure 5: The asymmetric Koch curve

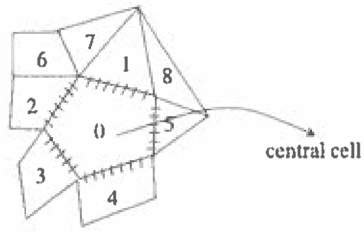


Figure 6: Section of soap bubble

Figure 7: Voronoi honeycomb for (a) a set of random points; (b) a set of random points with the constraint that they are no closer than a certain minimum distance.

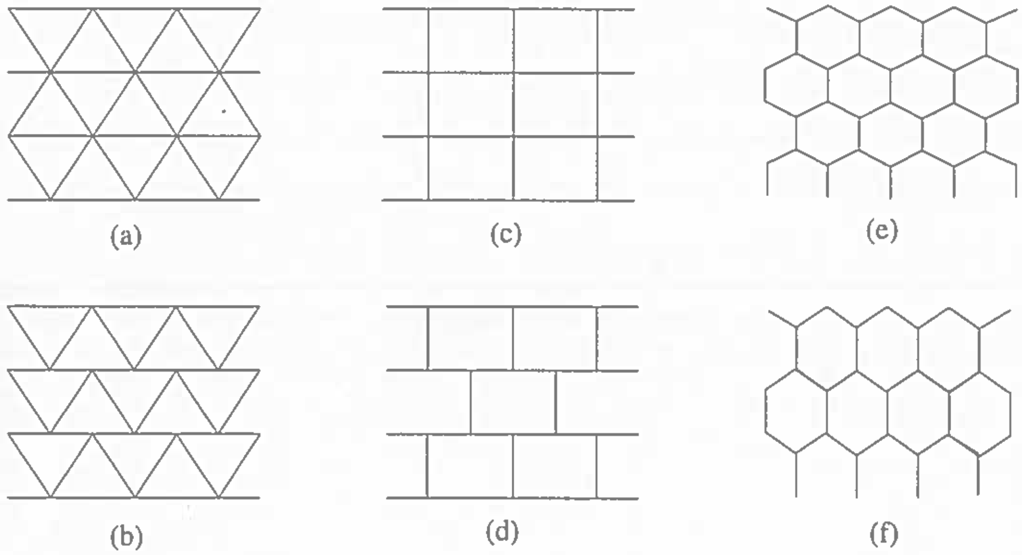


Figure 8: Packing of two-dimensional cells to fill a plane: (a, b) Two packings of equilateral triangles with $Z_e = 6$, and $Z_e = 4$, respectively. When $Z_e = 4$, $n = 4$, topologically. (c, d) Two packings of squares with $Z_e = 4$ and $Z_e = 3$, respectively. When $Z_e = 3$, $n = 6$ topologically. (e) Packing of regular hexagons. (f) Packing of irregular hexagons.

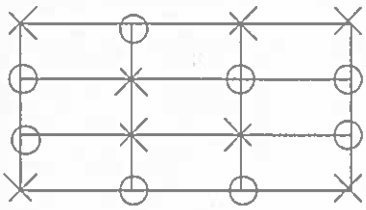


Figure 9:

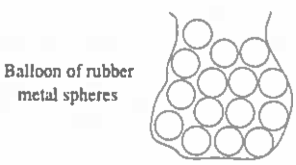
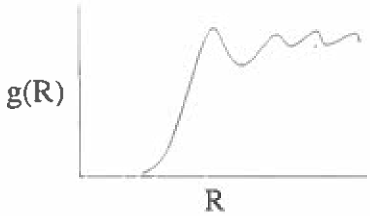


Figure 10:



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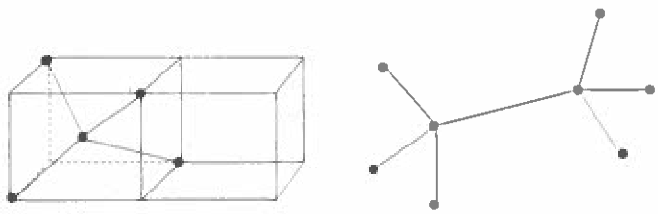

Figure 12:

Figure 13: Some of Budhurams sketches of the interiors of the cathedral.